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The Gravitational Field of a Nullicle†

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Abstract

We present a natural generalization of the Schwarzschild solution. It is an exact solution of the vacuum Einstein field equations and is interpreted as the gravitational field of a nullicle (null-particle).

1. *Introduction*

The problem of finding an exact solution of the vacuum Einstein field equations corresponding to a massless particle or nullicle has attracted some attention in recent years. In the linearized theory the fields of beams and pulses of light have been studied long ago by Tolman (1934) whose work has been extended into the realm of the exact theory by Bonnor (1969). Bonnor (1970a, 1970b) has extensively studied the gravitational fields of null fluids. More recently Aichelburg & Sexl (1971) have described a solution of the vacuum field equations for a massless particle which they obtain from the Schwarzschild solution using a Lorentz transformation. The Riemann tensor vanishes everywhere except on a null 3-surface which contains the null world-line of their massless particle. On the 3-surface the Riemann tensor becomes infinite. In this paper we adopt a different point of view.

When the Schwarzschild solution is written in the Kerr-Schild (1965) form we are provided with a natural background Minkowski space-time (cf. Section 2). The source of the solution may then be considered as being a time-like geodesic in this background space-time (Robinson & Trautman, 1962). In this paper we demonstrate that if one takes the same metric form but assumes that the source is a null geodesic in the background space-time then the metric is again an exact solution of the vacuum Einstein field equations. The field remains Petrov Type D but is singular not only on the world-line but also on a null 3-surface containing the world-line in the background space-time. The field (Riemann tensor) falls off smoothly to zero away from this 3-surface.

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2. *The Metric Form*

We begin by describing briefly how to construct the Kerr-Schild form of the Schwarzschild solution before proceeding to the null case.

Let $X^a = x^a(u)$ be a time-like world-line in Minkowski space-time with metric $\eta_{ab} = \text{diag}(1, 1, 1, -1)$ and having u as proper-time along it. Let it be the history of a particle of proper-mass m (= constant). The unit tangent to this world-line is denoted by $\lambda^a = dx^a/du$ so that $\eta_{ab}\lambda^a\lambda^b = -1$. If X^a are the coordinates of an event off the world-line and if $x^a(u)$ is the event of intersection of the past null cone with vertex X^a and the world-line, then the vector $\xi^a = X^a - x^a(u)$ is a null vector. Defining the (Lorentz) scalar $r = -\eta_{ab}\lambda^a\xi^b$ we easily see that r is positive and vanishes if and only if X^a lies on the world-line. It is called the 'retarded distance' of X^a from the world-line (cf. Synge, 1970). We can utilize this retarded construction to raise u, λ^a, $\mu^a = d\lambda^a/du$, r to the status of *retarded fields* on Minkowski space-time by parallel propagation along the generators of the future null-cones with vertices on the world-line (in the case of u we define $u(X) = u(x)$). Their derivatives are then readily evaluated and one finds (Synge, 1970)

$$u_{,a} = -r^{-1}\xi_a; \qquad \lambda_{a,b} = -r^{-1}\mu_a\xi_b \tag{2.1a}$$

$$\xi_{a,b} = \eta_{ab} + r^{-1}\lambda_a\xi_b; \qquad r_{,a} = -\lambda_a + B\xi_a \tag{2.1b}$$

where $B = (1 - W)r^{-1}$ and $W = -\mu_a\xi^a$. Defining the null vector $k^a = r^{-1}\xi^a$ we introduce the metric†

$$g_{ab} = \eta_{ab} + \frac{2m}{r}k_ak_b \tag{2.2}$$

This metric has Kerr-Schild form with k^a null with respect to g_{ab} and η_{ab}, the metric of the background Minkowski space-time. One can easily show, using the formulae (2.1), that (2.2) is a solution of the vacuum Einstein equations, $R_{ab} = 0$, provided $\mu^a = 0$, i.e. provided the world-line in the background space-time is a geodesic. Choosing θ, ϕ, r, u as coordinates one finds that in this case the metric (2.2) provides the line element

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - 2\,du\,dr - \left(1 - \frac{2m}{r}\right)du^2 \tag{2.3}$$

which is commonly called the Eddington form of the Schwarzschild solution.

We now take the metric (2.2) and assume that the world-line $X^a = x^a(u)$ is a null geodesic in the background space-time with u an affine parameter along it so that

$$\mu^a = 0, \qquad \eta_{ab}\lambda^a\lambda^b = 0 \tag{2.4}$$

† We choose units for which $c = G = 1$ throughout.

We shall refer to this world-line as the history of a nullicle. The constant m we shall take to be the energy of the nullicle by analogy with the Schwarzschild case above. The formulae (2.1) become simplified to read

$$u_{,a} = -r^{-1}\xi_a; \qquad \lambda_{a,b} = 0 \tag{2.5a}$$

$$\xi_{a,b} = \eta_{ab} + r^{-1}\lambda_a\xi_b; \qquad r_{,a} = -\lambda_a \tag{2.5b}$$

Since the scalar r is given by

$$r = -\eta_{ab}\lambda^a\xi^b = -\eta_{ab}\lambda^a X^b + \eta_{ab}\lambda^a x^b(u) \tag{2.6}$$

we see that it vanishes when $X^a = x^a(u)$, i.e. when the field event X^a lies on the world-line. However r also vanishes when X^a lies on the null 3-surface

$$\Sigma\colon \eta_{ab}\lambda^a X^b = c_0 \tag{2.7}$$

where $c_0 = \eta_{ab}\lambda^a x^b$, which is constant (independent of u) on account of (2.4). This null 3-surface has normal λ^a and contains the null geodesic $X^a = x^a(u)$ (see figure).

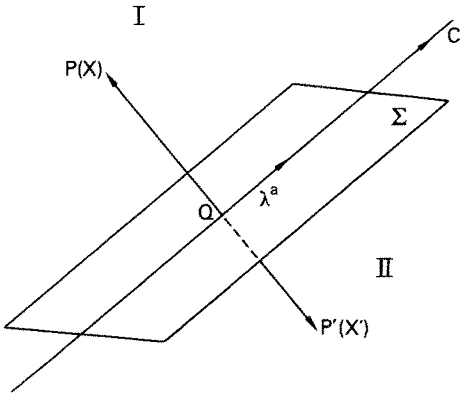

Figure 1.—The null world-line C in the background Minkowski space-time with the null 3-surface Σ containing C. The field events P and P' in regions I and II respectively are also shown. The lines QP and QP' are null.

We see from the figure that if the field event $P(X)$ lies in the region I of Minkowski space-time the retarded construction is valid. However moving to the opposite side of the null 3-surface Σ the retarded construction breaks down since the past null-cone with vertex at the field event $P'(X')$ will not intersect the world-line $X^a = x^a(u)$. In this region we must define *advanced fields* u, λ'^a, r' by parallel propagation along the generators of the past null-cones with vertices on the world-line (in the case of u we have $u(X') = u(x)$). In addition we have, in this case,

$$r' = \eta_{ab}\lambda'^a\xi'^b; \qquad \xi'^b = X'^b - x^b(u) \tag{2.8}$$

and so the formulae (2.5) become, in region II,

$$u_{,a} = r'^{-1}\xi'_a; \qquad \lambda'_{a,b} = 0 \tag{2.9a}$$

$$\xi'_{a,b} = \eta_{ab} - r'^{-1}\lambda'_a\xi'_b; \qquad r'_{,a} = \lambda'_a \tag{2.9b}$$

Inspecting the formulae (2.5) and (2.9) we notice that we may pass from (2.5) to (2.9) by formally changing the sign of r. Hence the metric (2.2) for our space-time becomes

$$g_{ab} = \eta_{ab} + \frac{2m}{|r|}k_a k_b, \qquad r \neq 0 \tag{2.10}$$

with $r > 0$ in region I and $r < 0$ in region II. It remains for us to check that (2.10), subject to the conditions (2.4) and the formulae (2.5) and (2.9) (remembering $k_a = r^{-1}\xi_a$) is an exact solution of the vacuum Einstein field equations.

3. *The Riemann Tensor and Field Equations*

We shall consider only the region $r > 0$. The metric (2.10) may thus be written

$$g_{ab} = \eta_{ab} + \gamma_{ab} \tag{3.1}$$

where

$$\gamma_{ab} = \frac{2m}{r}k_a k_b = \frac{2m}{r^3}\xi_a\xi_b \tag{3.2}$$

The Riemann tensor is

$$R_{abcd} = L_{abcd} + Q_{abcd} \tag{3.3}$$

where

$$L_{abcd} = \tfrac{1}{2}(\gamma_{ad,bc} + \gamma_{bc,ad} - \gamma_{ac,bd} - \gamma_{bd,ac}) \tag{3.4}$$

and

$$Q_{abcd} = g^{pq}([ad,p][bc,q] - [ac,p][bd,q]) \tag{3.5}$$

$$[ad,p] = \tfrac{1}{2}(\gamma_{pa,d} + \gamma_{dp,a} - \gamma_{ad,p}) \tag{3.6}$$

A straightforward but tedious calculation using the formulae (2.4) and (2.5) yields

$$\begin{aligned} L_{abcd} = {} & \frac{2m}{r^3}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) + \frac{3m}{r^4}\{\lambda_a(\xi_c\eta_{bd} - \xi_d\eta_{bc}) \\ & + \lambda_b(\xi_d\eta_{ac} - \xi_c\eta_{ad}) + \lambda_c(\xi_a\eta_{bd} - \xi_b\eta_{ad}) \\ & + \lambda_d(\xi_b\eta_{ac} - \xi_a\eta_{bc})\} + \frac{6m}{r^5}\{\xi_a\xi_d\lambda_b\lambda_c + \xi_b\xi_c\lambda_a\lambda_d \\ & - \xi_b\xi_d\lambda_a\lambda_c - \xi_a\xi_c\lambda_b\lambda_d\} \end{aligned} \tag{3.7}$$

and using the result

$$g^{ab} = \eta^{ab} - \frac{2m}{r^3}\xi^a\xi^b, \qquad \xi^a = \eta^{ab}\xi_b \tag{3.8}$$

we find

$$Q_{abcd} = \frac{2m^2}{r^6}\{\xi_a\xi_d\eta_{bc} + \xi_b\xi_c\eta_{ad} - \xi_a\xi_c\eta_{bd} - \xi_b\xi_d\eta_{ac}\} \tag{3.9}$$

We now calculate the Ricci tensor from

$$R_{bc} = g^{ad}R_{abcd} \tag{3.10}$$

using (3.8), and find that

$$R_{bc} = \eta^{ad}L_{abcd} \tag{3.11}$$

so that the non-linear part of the Ricci tensor vanishes for the metric (2.10). This appears to be a common property of solutions with null sources. Both Bonnor (1969) and Aichelburg & Sexl (1971) found their solutions to have this property. However it is certainly not confined to null sources for it is also true of the Schwarzschild solution (2.2). The important result of our work is, however, that the right-hand side of (3.11) also vanishes and thus the metric (2.10) is an exact solution of the vacuum field equations $R_{bc} = 0$ and may be interpreted as giving the gravitational field of a nullicle of energy m.

4. *Discussion*

The Riemann tensor for our solution, given by (3.3), (3.7) and (3.9), is of Petrov Type D and k^a is a principal null direction so that

$$R_{abc[d}k_{e]}k^b k^c = 0 \tag{4.1}$$

with the scalar products here calculated with the full metric g_{ab}. The Riemann tensor is singular on $r = 0$, i.e. on the null 3-surface in the background Minkowski space-time. From (3.7) and (3.9) we see that

$$L_{abcd} = O(r^{-3}), \qquad Q_{abcd} = O(r^{-4}) \tag{4.2}$$

and thus L_{abcd} is the dominant part of the Riemann tensor from large r away from $r = 0$. It is interesting to note that

$$L_{abc[d}k_{e]}k^b k^c = 0 \tag{4.3}$$

with the scalar products calculated with either of the metrics η_{ab} or g_{ab}. In addition

$$g_{ab}\lambda^a\lambda^b = \frac{2m}{r} > 0 \tag{4.4}$$

so that, as $r \to \infty$, λ^a becomes null with respect to the full metric g_{ab} and also we have

$$L_{abc[d}\lambda_{e]}\lambda^b\lambda^c = 0 \tag{4.5}$$

where the scalar products are calculated with η_{ab}. Hence we conclude from (4.3), (4.4) and (4.5) that k^a, λ^a are both principal null directions of the Riemann tensor for large values of r.

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